The adjacency operator of an infinite directed graph

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数理解析研究所講究録
第 743 巻 1991 年 93-103

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§ **1. Introduction.** In the graph theory, an adjacency matrix has been considered for finite graphs [2]. In [7], Mohar introduced the adjacency operator $A(G)$ for an infinite undirected graph G and discussed its spectral radius $r(G) = r(A(G))$. One of his main results is that if a sequence $\{G_n\}$ of subgraphs of a locally finite graph G with bounded valency converges to G, then $r(G_n)$ converges to $r(G)$. Recently Biggs, Mohar and Shawe-Taylor [1] discussed the relations between structure and the spectral radius of a undirected graph with a finite isoperimetric constant. Since a graph discussed by them is undirected, if its adjacency operator is bounded, then it is self-adjoint. From this point, we defined the adjacency operator for an infinite directed graph in [4], in which the adjacency operator is not always self-adjoint even if it is bounded.

This report consists of 5 sections;

§ 1. Introduction.

§ 2. Adjacency operators.

§ 3. Classifications by adjacency operators.

§ 4. Convergence of graphs.

§ 5. The spectrum of a graph.

In § 2, we memtion some basic definitions on graphs and the definition of the adjecncy the adjacency operator $A(G)$ of an infinite directed graph G. In § 3, several classes of adjacency operators are characterized by their graphs. For example, $A(G)$ is normal, hyponormal, unitary and positive etc.. In § 4, we introduce the numerical radius of a graph and discuss its continuity. In the final section, we consider the form of the spectrum of a graph.

§ **2. Adjacency operators.** First we state some definitions for a graph. A directed graph $G = (V, E, \partial^+, \partial^-)$ is a system of sets V, E and maps $\partial^\pm : E \to V$. An element

$v \in V$ (resp. $e \in E$) is called a vertex (resp. arc). For an arc $e \in E, \partial^+(e) \in V$ is an initial vertex and $\partial^-(e) \in V$ is a terminal vertex. For each vertex $v \in V$, the outdegree $d^+(v)$, the indegree $d^-(v)$ and the valency (or degree) $d(v)$ are defined by

$$d^+(v) = \sharp\{e \in E; \partial^+(e) = v\}, \quad d^-(v) = \sharp\{e \in E; \partial^-(e) = v\},$$
$$\text{and} \quad d(v) = d^+(v) + d^-(v),$$

respectively. A graph is called locally finite if every vertex has finite valency. A graph has bounded valency if there is a constant $M > 0$ such that $d(v) \leq M$ for any vertex $v \in V$. We introduce common servers and receivers for pairs of vertices. If $\partial^+(e) = u$ and $\partial^-(e) = v$ for some $e \in E$, then u is a server of v, and v is a receiver of u. A vertex w is called a common server of u and v, if w is a server of u and v. Similarly w is called a common receiver of u and v, if w is a receiver of u and v. Denote the number of all common servers (resp. common receivers) of u and v by $d^+(u,v)$ (resp. $d^-(u,v)$). We define the following subsets of V;

$$D^+(v) = \{u \in V; u \text{ is a receiver of } v\},$$
$$D^-(v) = \{u \in V; u \text{ is a server of } v\},$$
$$D^+(u,v) = \{w \in V; w \text{ is a common receiver of } u \text{ and } v\}, \text{and}$$
$$D^-(u,v) = \{w \in V; w \text{ is a common server of } u \text{ and } v\}.$$

Throughout this note, a graph means a locally finite directed graph without multiple arcs, that is, for any vertices $u, v \in V$ there exists at most one arc $e \in E$ with $\partial^+(e) = u$ and $\partial^-(e) = v$.

Next we define the adjacency operator of an infinite directed graph. Let H be a Hilbert space $\ell^2(V)$ with the canonical basis $\{e_v; v \in V\}$ defined by $e_v(u) = \delta_{v,u}$ for $u, v \in V$, and H_0 the linear span of $\{e_v; v \in V\}$. Now we consider linear operators A_0 and B_0 on H with the dense domain $Dom(A_0) = H_0 = Dom(B_0)$ defined by

$$A_0\left(\sum_{v\in V} x_v e_v\right) = \sum_{u\in V}\sum_{v\in D^-(u)} x_v e_u, \quad \text{and} \quad B_0\left(\sum_{v\in V} x_v e_v\right) = \sum_{u\in V}\sum_{v\in D^+(u)} x_v e_u$$

for $\sum_{v\in V} x_v e_v \in H_0$. Since G is locally finite, A_0 and B_0 are well-defined. Both operators are closable and $A_0^* \supset \overline{B_0}, B_0^* \supset \overline{A_0}$, where the bar denotes the closure.

Let us define a closed operator $A = A(G)$ with the domain $Dom(A)$ by

$$Dom(A) = \{x = \sum_{v\in V} x_v e_v \in H; \sum_{u\in V} \mid \sum_{v\in D^-(u)} x_v \mid^2 < \infty\}$$

and

$$Ax = \sum_{u\in V}\sum_{v\in D^-(u)} x_v e_u,$$

for $x \in Dom(A)$. We call $A = A(G)$ the adjacency operator of G. Here we remark that the above definition of $A(G)$ is the transpose of the usual one of G. Then we see that $A \supset \overline{A_0} = A_0^{**}$. Similarly we shall define a closed operator B with the domain $Dom(B)$ by

$$Dom(B) = \{x = \sum_{v\in V} x_v e_v \in H; \sum_{u\in V} \mid \sum_{v\in D^+(u)} x_v \mid^2 < \infty\}$$

and

$$Bx = \sum_{u\in V}\sum_{v\in D^+(u)} x_v e_u$$

for $x \in Dom(B)$.

LEMMA 2-1. *Let A be the adjacency operator of G. Then*

(1) $(Ae_v \mid e_u) = \begin{cases} 1 & \textit{if} \quad u \in D^+(v), \\ 0 & \textit{if not}, \end{cases}$ (2) $(A^*e_v \mid e_u) = \begin{cases} 1 & \textit{if} \quad u \in D^-(v), \\ 0 & \textit{if not}, \end{cases}$

(3) $(A^*Ae_v \mid e_u) = d^+(u, v)$, (4) $(AA^*e_v \mid e_u) = d^-(u, v)$,

(5) $\| Ae_v \| = \sqrt{d^+(v)}$, (6) $\| A^*e_v \| = \sqrt{d^-(v)}$.

We shall consider a necessary and sufficient condition for adjacency operators to be bounded and give an upper-bound for the norm. To do this, we put the maximal outdegree and indegree of G by

$$k^+ = k^+(G) = max\{d^+(v); v \in V\}, \text{and}$$
$$k^- = k^-(G) = max\{d^-(v); v \in V\}.$$

We sometimes regard E as a subset $V \times V$, that is, an arc $e \in E$ with $\partial^+(e) = u$ and $\partial^-(e) = v$ might be denoted by (u, v).

THEOREM 2-2. *Let A be the adjacency operator of a graph G.*

(1) *A is bounded if and only if G has bounded valency. Moreover in this case, $A = B^*, B = A^*$ and*

$$\| A \| \leq \sqrt{k^- k^+}.$$

(2) *Assume that G has bounded valency . If there exist k^- vertices $\{v_1, \ldots, v_{k^-}\}$ and k^+ vertices $\{u_1, \ldots, u_{k^+}\}$ such that $(v_i, u_j) \in E$ for $i = 1, \ldots, k^-, j = 1, \ldots k^+$, then*

$$\| A \| = \sqrt{k^- k^+}$$

§ **3. Classifications by adjacency operators.** We shall classify graphs with bounded valency by their adjacency operators. A source of a directed graph G is a vertex v whose $d^-(v) = 0$. A source v is called non-trivial if $d^+(v) \neq 0$. A sink of G is a vertex v whose $d^+(v) = 0$. A sink v is called non-trivial if $d^-(v) \neq 0$. And a graph G is normally symmetric if $d^-(u, v) = d^+(u, v)$ for any $u, v \in V$.

It is obvious that the adjacency A is self-adjoint if and only if the graph is undirected in the sense that $(u, v) \in E$ if $(v, u) \in E$.

THEOREM 3-1. *Let A be an adjacency operator of a graph G. Then*

(1) A is normal if and only if the graph G is normally symmetric.

(2) If A is hyponormal, then there does not exist a non-trivial sink of G.

(3) A is compact if and only if G has only finitely many arcs.

REMARK. *As in the above (2), if an adjacency operator A is co-hyponormal, then there dose not exist a non-trivial source of G.*

EXAMPLES: The above graph theoretical classification leads us the example of a normal operator in Fig.1. For this example, Fig.2 gives us an example whose adjacency operators is nonnormal and hyponormal.

It follows from Lemma 2-1 (3) and (4) that A is hyponormal if and only if the operator given by infinite matrix $(d^+(u,v) - d^-(u,v))_{u,v}$ is positive. Hence if A is hyponormal, then

$$d^+(u) \geq d^-(u) \quad \text{for all} \quad u \in V. \tag{0}$$

Clearly (0) implies G does not have a non-trivial sink. However the condition (0) does not imply the hyponormality of A. An example of this is posed by Fig.3. As a matter of fact, A is expressed as a matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Take a vector $x = {}^t(1,0,0,1)$. Then we have

$$((A^*A - AA^*)x \mid x) = -2,$$

so that A is not hyponormal. Furthermore we know that A is normaloid, i.e, $\| A \| = r(A)$, the spectral radius of A, whose related results will be considered after.

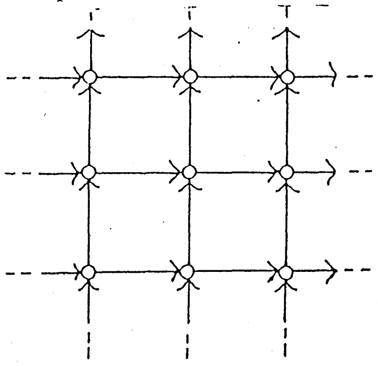


Fig.1

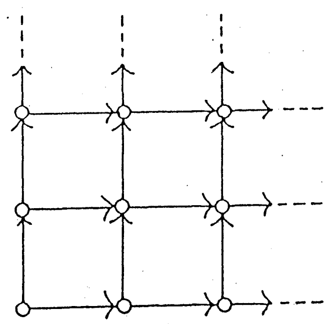


Fig.2

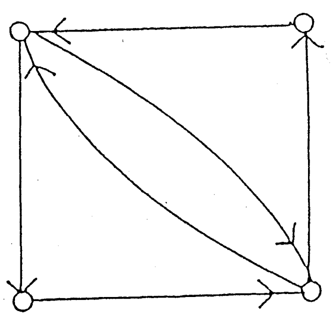


Fig.3

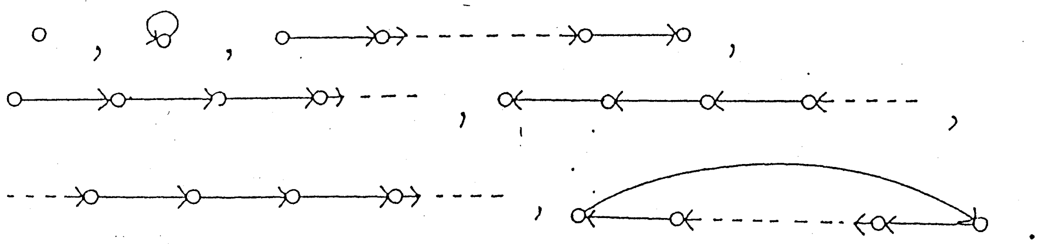
THEOREM 3-2. *Let A be an adjacency operator of a graph G.*

(1) The following are equivalent.

i) A is a partial isometry.

ii) For any vertex $v \in V, d^+(v) \leq 1$ and $d^-(v) \leq 1$.

iii) The connected components of G are one of the following,

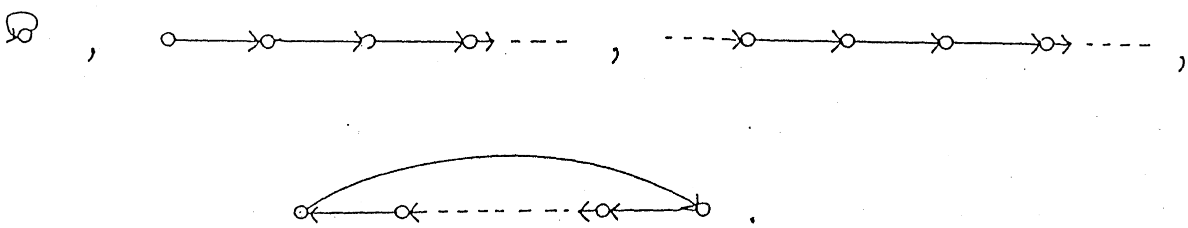


(2) The following are equivalent.

i) A is an isometry.

ii) For any vertex $v \in V, d^+(v) = 1$ and $d^-(v) \leq 1$.

iii) The connected components of G are one of the following,

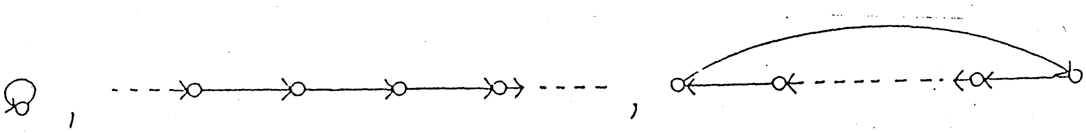


(3). The following are equivalent.

i) A is unitary.

ii) For any vertex $v \in V, d^+(v) = 1$ and $d^-(v) = 1$.

iii) The connected components of G are one of the following,



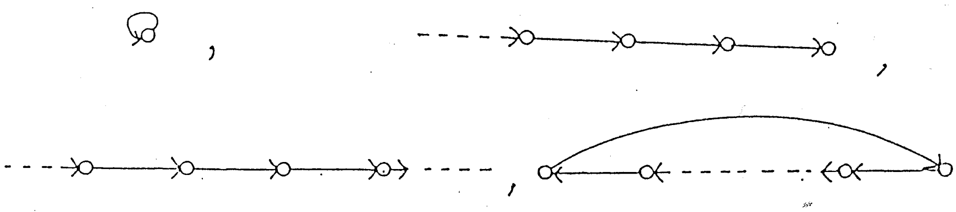
(4) A is a projection if and only if the connected components of G are one of the following,

REMARK. *As in the case of isometries, the following are equivalent.*

i) A is a co-isometry.

ii) For any vertex $v \in V, d^+(v) \leq 1$ and $d^-(v) = 1$.

iii) The connected components of a graph are one of the following,



In a directed graph, any sequence of consecutive arcs is called a walk. A walk is called a trail if all its arcs are distinct. Especially a trail whose endvertices coincide is called a circuit. Let $N_k(i,j)$ denote the number of walks of length k starting at vertex j and terminating at vertex i. If we denote $A^k = a_{ij}^{(k)}$, then it is known that $N_k(i,j) = a_{ij}^{(k)}$.[2,Theorem 1.9] We can characterize nilpotent operators by the existence of circuits.

THEOREM 3-3. *Let G be a finite graph and A be a non-zero adjacency operator.*

(1) A is nilpotent if and only if G has no circuits.

(2) If A is idempotent, then G has at least a loop.

A graph G is called trivial if G has no arcs. A simple undirected graph in which every pair of distinct vertices are adjacent is called a complete graph. A simple undirected graph in which if every pair of (not necessarily distinct) vertices are adjacent is a super complete graph.

THEOREM 3-4. *Let A be an adjacency operator of a graph G. A is positive if and only if the connected components of G are finite super complete or trivial.*

§ **4. Convergence of graphs.** In [7], one of his main result is that if a sequence $\{G_n\}$ of subgraphs of a graph G converges to G, then $r(G_n)$ converges to $r(G)$. But Mohar's result does not hold for infinite directed graphs. For example, we consider the shift graph P, whose adjacency operator is a unilateral shift, and the path P_n with length n as a subgraph of P. Though $r(P_n) = 0$ for any n by $A(P_n)^n = 0$, $r(P) = 1$. We note that, as an adjacency operator A is Hermitian in his case, the spectral radius $r(A)$ of A coincides with the numerical radius $w(A)$ of A. Here $w(T)$ of an operator T on a Hilbert space H is defined by

$$w(T) = \sup\{| (Tx, x) |; \| x \| = 1, x \in H\},$$

cf.[6]. So we call $w(G) = w(A(G))$ the *numerical radius* of G. By recent work in [3] and [5], we know that $w(P_n) = \cos \frac{\pi}{n+1}$ and so $w(P_n)$ converges to $1 = w(P)$. However we remark that the numerical radius of operators is not continuous with respect to the strong operator topology in [6:Prob220], whose counterexample is also acceptable for the numerical radius of graphs.

For another simple example, let E_n be the projection onto the subspace spanned by $\{e_k; k \geq n\}$. Then E_n converges to 0 strongly and $w(E_n) = 1$ for all n. As a matter of fact, E_n is regarded as the adjacency operator of the graph whose vertices are $\{1, 2, \dots\}$ and vertex k has only self-loop for $k \geq n$.

Nevertheless, we have the following result by assuming a bounded condition, which is known by the lower semicontinuity.

LEMMA 4-1. *Let T_n and T be operators on H.*

(1) *If $w(T_n) \leq w(T)$ for all n and T_n converges to T in the weak operator topology, then $w(T_n)$ converges to $w(T)$.*

(2) *If $\| T_n \| \leq \| T \|$ for all n and T_n converges to T in the strong operator topology, then $\| T_n \|$ converges to $\| T \|$.*

Next we difined the convergence of graphs. For $u, v \in V$, we denote $(u, v) \in E$ if there is an arc $e \in E$ such that $\partial^+(e) = u$ and $\partial^-(e) = v$. Let $\{G_n\}$ be a sequence of graphs and G a graph. We may assume that $V(G_n) = V(G)$ for all n without loss generality. Then G_n converges to G, in symbol, $G_n \to G \quad (n \to \infty)$ if for any vertices $u, v \in V(G)$ there exists a number N such that for all $n \geq N$, $(u, v) \in E(G)$ if and only if $(u, v) \in E(G_n)$. It means the convergence of all entries of the adjacencey operator, i.e. $(A(G_n))_{u,v} \to (A(G))_{u,v}$ for any $u, v \in V(G)$. We have the following generalization of Mohar's result.[7:Prop 4.2]

THEOREM 4-2. *Let $\{G_n\}$ be a sequence of subgraphs of a graph G. Then the following conditions are equivalent:*

(i) *G_n converges to G.*

(ii) *$A(G_n)$ converges to $A(G)$ in the strong operator topology.*

(iii) *$A(G_n)$ converges to $A(G)$ in the weak operator topology.*

For $x = (x_v) \in \ell^2(V)$, we denote $x \geq 0$ if $x_v \geq 0$ for all v and $| x | = (| x_v |)$.

LEMMA 4-3. *For a graph G,*

$$\begin{aligned} w(G) &= \sup\{(A(G)x, x); \| x \| = 1, x \geq 0\} \\ &= \sup\{(A(G)y, y); \| y \| = 1, y = \sum_{v \in W} y_v e_v \geq 0 \text{ and } W \text{ is finite }\}. \end{aligned}$$

From the graph theoretical view, the bounded condition in Lemma 4-1 is very natural.

COROLLARY 4-4. *If F is a subgraph of a graph G, then $w(F) \leq w(G)$.*

Consequently we have a generalization of a result by Mohar [7].

THEOREM 4-5. *Let $\{G_n\}$ be a sequence of subgraphs of a graph G. If G_n converges to G, then $w(G_n)$ converges to $w(G)$.*

COROLLARY 4-6. *For a graph G,*

$$\| A(G) \| = \sup\{\| A(F) \|; F \text{ is a finite subgraph of } G\}.$$

Corollary 4-7. If F is a subgraph of a graph G, then $\| A(F) \| \leq \| A(G) \|$.

Theorem 4-8. *Let $\{G_n\}$ be a sequence of subgraphs of a graph G. If G_n converges to G, then $\| A(G_n) \|$ converges to $\| A(G) \|$.*

Remark 4-9. If G is an undirected graph, then $r(G) = \sup\{r(F); F$ is a finite subgraph of $G.\}$ by [7]. To the contrary, if G is a directed graph, then it is not true, e.g. a shift graph because the adjacency operator of its finite subgraph is nilpotent. However since $r(G) = \lim_{n\to\infty} \| A(G)^n \|^{\frac{1}{n}}$, one can prove that if F is a subgraph of a graph G, then $r(G) \geq r(F)$.

§ 5. **The spectrum of a graph.** In this section, we discuss relations between properties of a graph and its spectrum.

Theorem 5-1. *Let G be a infinite graph. Then the spectra of G is symmetric with respect to real axis.*

A graph is a bipartite graph if the vertices of G can be partitioned into two disjoint sets V_1 and V_2 in such a way that every edge has one vertex in V_1 and one vertex in V_2.

Theorem 5-2. *Let G be a bipartite graph. Then the spectra is symmetric with respect to zero.*

Next, we define the isoperimetric constant $i(G)$ of a graph G. For a graph G and a finite subset X of the vertices of G, we define ∂X to be the subset of arcs of G incident with exactly one vertex of X.

$$i(G) = \inf\{\frac{|\partial X|}{|X|}; X \text{ is a finite subset of } V(G)\}$$

A graph is a k-semiregular graph if there exists a constant k such that $d^-(v) = k$ or $d^+(v) = k$ for any $v \in V$.

LEMMA 5-3. *If G is an infinite k-semiregular graph such that $i(G)=0$, then $r(G)\geq k$.*

THOREM 5-4. *If G is an infinite graph such that $i(G)=0$, then*

$$\max\{\ell^-,\ell^+\}\leq r(G)$$

where ℓ^- (resp. ℓ^+) is a minimal number of indegree (resp. outdegree)of G.

COROLLARY 5-5. *If G is a k-semiregular graph such that $i(G)=0$ and $k^-=k^+=k$, then $A(G)$ is normaloid and $r(G)=k$.*

References.

[1] N.L.Biggs, B.Mohar and J. Shawe-Taylor: The spectral radius of infinite graphs, Bull.London Math.Soc., 20 (1988), 116-120.

[2] D.M.Cvetković , M.Dood, and H.Sachs: Spectra of Graphs, VEB Deutscer Verlag d.Wiss., Berlin, 1980 ; , Academic, New York, 1980.

[3] K. R. Davidson and J. A. R. Holbrook: Numerical Radii of zero-one matrices, Michigan Math. J., 35 (1988), 261-267.

[4] M. Fujii, H. Sasaoka and Y. Watatani: Adjacency operators of infinite directed graphs, Math.Japon., 34 (1989), 727-735.

[5] U. Haagerup and P. de la Harpe: The numerical radius of a nilpotent operator on a Hilbert space, preprint.

[6] P. R. Halmos: A Hilbert Space Problem Book, 2nd Ed., Springer, New York, 1982

[7] B. Mohar :The spectrum of an infinite graph, Linear Algebra Appl.,48 (1982), 245-256

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